On the equiconvergence of orthogonal series

(Article without co-authors in the collection of scientific papers “Mathematical methods and software tools in scientific researches”, preprint of the Institute of Cybernetics of the National Academy of Sciences of Ukraine)

In practice, approximation methods are most frequently used for functions given over finite points t_1, t_2, …, t_n (all t_i belong to a certain given approximation interval $[a, b]$). Hereby, the most frequent method is the least-squares method both classical and modified in many ways: with weighing factors, on a non-regular mesh, etc.

On the other hand, from the theoretical perspective, approximation methods, which use approximee $f(t)$ continual information, including any $f(t)$ values over the interval $[a, b]$ are investigated to the fullest extent possible. That is why an idea naturally arises to compare function reconstruction methods based on discrete and continual information. Hereby, equiconvergence theorems proposed in this paper are of great importance. We shall note that equiconvergence investigations carried out for different orthogonal series have the deep history (see e. g. [1]) and are still being continued [2], [3].

The prime object of this paper are theorems of equiconvergence of the Fourier-Jacobi series and Fourier-Hahn series. However, theoretical constructions giving these outcomes are more commonly used and thus are of interest by themselves.

1. Integral expression for the difference of Fourier partial sums

We shall specify required definitions. Let $\mu(t)$ and $\nu(t)$ be nonnegative non-decreasing, not identically constant functions defined over the finite interval $[a, b]$. Hereinafter, such functions will be called measures (being aware of a difference of the introduced functions with classical measures). We will suppose that

$$\int_a^b d\mu(t) = \int_a^b d\nu(t) = 1 \tag{1}$$

Here and elsewhere, integrals shall be understood in a sense of Lebesgue-Stieltjes.

Let us introduce dot products $(f, g)_\mu = \int_a^b f(t)\cdot g(t)d\mu(t)$ and $(f, g)_\nu = \int_a^b f(t)\cdot g(t)d\nu(t)$. Let $L_{2\mu}$ and $L_{2\nu}$ be spaces of functions with norms $\|f\|_{2\mu} = ((f,f)_\mu)^{1/2}$ and $\|f\|_{2\nu} = ((f,f)_\nu)^{1/2}$, correspondingly. Let us set $L_{\mu\nu} = L_{2\mu} \cap L_{2\nu}$. It follows from the condition (1) that the set $L_{\mu\nu}$ is not empty, since it includes at least a set of algebraic polynomials (as well as all continuous functions). Next we suppose that $L_{\mu\nu}$ is densely in $L_{2\mu}$.

Let $\{p_n(t)\}$ and $\{q_n(t)\}$ be sequences of generalized polynomials, which are orthonormalized regarding corresponding dot products. It is hereby supposed that the specidfied systems of generalized polynomials are obtained by means of the orthogonalization of the same system of linearly-independent functions $\{\varphi_n(t)\}$, $\forall n \quad \varphi_n \in L_{\mu\nu}$ (e. g., $\varphi_n(t) = t^n$). Then it is obvious that $\forall n \quad p_n \in L_{2\nu}$ and $\forall n \quad q_n \in L_{2\mu}$.

For $\forall f \in L_{2\mu}$, let us denote by $S_{n\mu}(f;t)$ the n-order Fourier sum of function f expansion into series using the system $\{p_n\}$, i. e.

$$S_{n\mu}(f;t) = \sum_{k=0}^{n} c_k p_k(t), \quad c_k = (f, p_k)_\mu .$$

Next let us denote

$$r_{n\mu}(f;t) = f(t) - S_{n\mu}(f;t); \quad K_{n\mu}(t,x) = \sum_{k=0}^{n} p_k(t) p_k(x).$$

Then

$$S_{n\mu}(f;t) = \int_a^b K_{n\mu}(t;x) f(x) d\mu(x).$$

$S_{n\nu}(f;t)$, $r_{n\nu}(f;t)$ и $K_{n\nu}(t,x)$ have the similar meaning. Finally, for $\forall f \in L_{\mu\nu}$ let us set $R_n(f;t) = S_{n\nu}(f;t) - S_{n\mu}(f;t)$. the following statement is valid.

Lemma 1. Let $f \in L_{\mu\nu}$. Then

$$R_n(f;t) = \int_a^b K_{n\nu}(t;x) r_{n\mu}(f;x) d\nu(x) = \int_a^b K_{n\mu}(t;x) r_{n\nu}(f;x) d\mu(x). \tag{2}$$

Proof. Since the sequences of the orthonormalized polynomials $\{p_n\}$ and $\{q_n\}$ were obtained from the same system of functions $\{\varphi_n\}$, for the random generalized polynomial $Q_k(t) = \sum_{i=0}^{k} a_i \varphi_i(t)$ the equation ($k \le n$) is valid

$$S_{n\mu}(Q_k;t) = S_{n\nu}(Q_k;t) = Q_k(t). \tag{3}$$

Then, for random n $S_{n\mu}(f;t) = S_{n\nu}(S_{n\mu}(f);t)$. From this equation and the operator linear property determined by the Fourier partial sum we obtain

$$R_n(f;t) = S_{n\nu}(f;t) - S_{n\nu}(S_{n\mu}(f);t) = S_{n\nu}(f - S_{n\mu}(f);t) =$$

$$= S_{n\nu}(r_{n\mu}(f);t) = \int_a^b K_{n\nu}(t;x) r_{n\mu}(f;x) d\nu(x).$$

Similarly, the second part of the equation (2) is proved. The lemma is proved.

We shall note that the similar result was proved in [2], but for harder constraints over the function $\mu(t)$ and $\nu(t)$. In particular, if $S_{n\mu}(f;t)$ converges pointwise to the function $f(t)$, in (2), $r_{n\mu}(f;x)$ shall be replaced with the sum $\sum_{k=n+1}^{\infty} (f, p_k)_\mu p_k(x)$, and this is in good agreement with the article outcome [2].

The proved lemma implies as follows.

Theorem 1. Let $f \in L_{\mu\nu}$. Then

$$\mathop{Sup}_{\substack{f\in L_{\mu\nu}\\ \|r_{n\mu}(f)\|_{2\nu}>0}} \frac{|R_n(f;t)|}{\|r_{n\mu}(f)\|_{2\nu}} \le \sqrt{\sum_{k=0}^{n} q_k^2(t)}. \tag{4}$$

$$\mathop{Sup}_{\substack{f\in L_{\mu\nu}\\ \|r_{n\nu}(f)\|_{2\mu}>0}} \frac{|R_n(f;t)|}{\|r_{n\nu}(f)\|_{2\mu}} \le \sqrt{\sum_{k=0}^{n} p_k^2(t)}. \tag{5}$$

Proof. Let us apply the Cauchy-Bunyakovsky inequality for the first part of formula (2)

$$|R_n(f;t)| \le \|K_{n\nu}(t,\cdot)\|_{2\nu} \cdot \|r_{n\mu}(f)\|_{2\nu}.$$

Then, using the property of orthogonality the polynomial system $\{q_n\}$, it is easy to show that

$$\|K_{n\nu}(t,\cdot)\|_{2\nu}^2 = \sum_{k=0}^{n} q_k^2(t).$$

Thus, the first theorem proposition is proved. Similarly, the inequality (5) is proved. The theorem is proved.

Let us draw another outcome following from the lemma 1. For this, we will denote by $L_{r\gamma}$ $(1 \le r \le \infty)$ the spaces of functions with the norm

$$\|f\|_{r\gamma} = \left(\int_a^b |f(t)|^r \, d\gamma(t) \right)^{1/r}$$

and $L_{\infty\gamma}$ — the space with the norm

$$\|f\|_{\infty\gamma} = \operatorname*{vrai\,sup}_{t\in[a,b]} |f(t)| = \inf_{e:\gamma(e)=0} \sup_{t\in[a,b]\setminus e} |f(t)|.$$

Theorem 2. The inequality $(1 \le r \le \infty)$ is valid

$$\mathop{Sup}_{\substack{f\in L_{r\mu}\cap L_{r\nu}\\ \|r_{n\mu}(f)\|_{r\nu}>0}} \frac{\|R_n(f)\|_{r\nu}}{\|r_{n\mu}(f)\|_{r\nu}} \le \|K_{n\nu}\|_{r\nu}.$$

In addition, a similar inequality is true, if we interchange μ and ν.

The proof of the theorem follows from the lemma 1 and the definition of a norm in the corresponding space.

We shall note that the proved theorems are very general. Below, we will consider their application in particular situations.

2. Description of class $L_{2\nu}$ in case of discrete measure ν.

Let μ and ν be a continuous and a discrete measure over the interval $[a, b]$ correspondingly, and generally as the measure ν any member of discrete measures of the sequence $\{\nu_N\}$, can be, $N \in \mathbb{N}$. Supports of these measures are finite sets of points $T_N = \{t_{iN},\ i = \overline{0, N},\ t_{iN} \in [a, b]\}$. Let us suppose that the functions $\nu_N(t)$ are left-continuous, and the set $T_\nu = \bigcup_N T_N$ is always densely in $[a, b]$. In addition, it is quite natural to consider the situation, when the sequence of measures $\nu_N(t)$ converges in any metric to the measure $\mu(t)$ at $N \to \infty$. That is why the fulfillment of the following condition is also assumed

$$\rho(\mu, \nu_N) = \max_t |\mu(t) - \nu_N(t)| \to 0 \qquad \text{at } N \to \infty. \tag{6}$$

Let us provide a specific example. Let $[a, b] = [-1, 1]$, $d\mu(t) = \omega_{\alpha\beta}(t)dt = (1 - t)^\alpha(1 + t)^\beta dt$, $\alpha, \beta > -1$ (i. e. $\omega_{\alpha\beta}(t)$ is the Jacobi weight function). As discrete measures $\nu_N(t)$, we will select measures determined by the equation: $d\nu_N(t) = \sum_{i=0}^N \beta_{iN}\delta(t - t_{iN})dt$, where β_{iN} are weighting factors for Hahn polynomials [4], reduced to the integral [–1, 1] and considering the equation (1), $t_{iN} = -1 + 2i/N$ ($i = \overline{0, N}$), $\delta(t)$ is the delta function. The paper [5] proves that the condition (6) is fulfilled in this situation.

Let us consider what can be class $L_{2\nu}$ if the measure ν is discrete. First of all, it is obvious that this class includes functions taking finite values and, for definiteness, left-continuous over the set T_ν. In addition, for these functions the sequence of values $\|f\|_{2\nu_N}$ is collectively restricted; hereby, taking into account the condition (6), it is natural to try taking the value $\lim_N \|f\|_{2\nu_N}$ as a norm of the function f in the space $L_{2\nu}$, i. e. to set $\|f\|_{2\nu} = \lim_N \|f\|_{2\nu_N}$. It is easily seen that this value is not the norm (it will be a seminorm), since the condition $\|f\|_{2\nu} = 0$ does not generally imply $f(t) = 0$.

Based on the set $L_{2\nu}$, a "full-fledged" space of functions can be organized by two methods. First, we can introduce equivalence classes, similar to spaces L_p. Second, we can restrict ourselves with functions, for which the zero equivalence over the set T_ν (recall that T_ν is always densely in $[a, b]$) will be a sufficiency condition of the zero equivalence over the whole set $[a, b]$. It is easy to show that, to execute the second method, it is sufficient to set $L_{2\nu} = C[a, b]$. Then $\forall f \in L_{\mu\nu}$ $\|f\|_{2\nu} = \|f\|_{2\mu}$, but, naturally, in this case the set $L_{2\nu}$ will be incomplete regarding the norm $\|\cdot\|_{2\nu}$.

Let us consider the first method. But before we will estimate the difference module

$$\Delta_{\mu\nu}(f) = \left| \int_a^b f(t)d\mu(t) - \int_a^b f(t)d\nu(t) \right|$$

for a bounded function $f(t)$. If no restrictions, except of the above ones, are set for the function f, we have the inequality [6], which is unimprovable over the class $L_{\mu\nu}$:

$$\Delta_{\mu\nu}(f) \le \max_{t\in[a,b]} |f(t)| \cdot \operatorname*{Var}_a^b(\mu - \nu),$$

where $\operatorname*{Var}_a^b(g)$ is the variation of the function $g(t)$ over the interval $[a, b]$. But, as it can be shown by the example of the Jacobi and Hahn weight functions, the difference variation $\mu(t) - \nu(t)$ does not need to tend to zero, even if $\rho(\mu - \nu) \to 0$. That is why let us also require that the function f has a bounded variation over $[a, b]$. Then, using the formula of integration by parts and the equation (1), we obtain

$$\Delta_{\mu\nu}(f) \le \max_t |\mu - \nu| \cdot \operatorname*{Var}_a^b(f) \tag{7}$$

And this assessment tends to zero, if $\rho(\mu - \nu) \to 0$. From this a requirement follows: the functions from the class $L_{2\nu}$ shall have the bounded variation. Then $L_{2\nu} \subset L_{2\mu}$, $L_{\mu\nu} = L_{2\nu}$ and $\forall f \in L_{2\nu}$ $\|f\|_{2\nu} = \|f\|_{2\mu}$, and any function from the space $L_{2\nu}$ can be considered as a representative of a some equivalence class from the space $L_{2\mu}$. Below, we will use $L_{2\nu}$ to denote the class of functions, which are left-continuous and have the finite variation over $[a, b]$.

The class of functions $L_{2\nu}$, which is determined in such a way, allow proving of an important statement: since $L_{2\nu} \subset L_{2\mu}$ and $L_{2\nu}$ is densely in $L_{2\mu}$, we can equate the inequality (5), and it is easy to show. In other words, the assessment (5) is unimprovable over the class $L_{\mu\nu}$.

3. Comparison theorems for the Fourier-Jacobi series and Fourier-Hahn series

Let us investigate the equiconvergence of the Fourier-Jacobi series and Fourier-Hahn series. As above, we will denote their corresponding measures by $\mu(t)$ and $\nu_N(t)$. Each of these measures is characterized by two parameters α and β. We will further consider only a case, when parameters of these measures $\mu(t)$ and $\nu_N(t)$ coincide. Since the measure $\nu_N(t)$ depends on the parameter N, we will introduce an index N for the difference $R_n(f;t) = R_{nN}(f;t)$ and polynomials $q_n(t) = q_{nN}(t)$, generated by the measure ν_N.

The theorem 1 gives two assessments of the difference R_{nN}. In both assessments, the functions $r_{n\mu}$ and $r_{n\nu_N}$ are assessed by “foreign” norms. They can be reduced to the norms of the spaces $L_{2\mu}$ and $L_{2\nu_N}$, correspondingly, using the inequality (7). In this case, the question of the finiteness of variations of functions $r_{n\mu}$ and $r_{n\nu_N}$ at $N \to \infty$ arises. This issue shall be investigated individually: we shall note that generally even the uniform zero convergence of bounded-variation functions does not by itself guarantee the uniform boundedness of their variation. That is why we will here prove only that the variations of the above functions cannot increase faster than $O(n)$.

First of all, we will show that for an arbitrary polynomial Q_n of n degree its variation over the interval $[-1,1]$ does not exceed $\pi n M_n$, where $M_n = \max_{t\in[-1,1]} |Q_n|$, i. e.

$$\operatorname*{Var}_{-1}^{1}(Q_n) \le \pi n M_n \,. \tag{8}$$

This proposal follows from the Bernstein inequality [7]: $\forall t \in (-1,1)$ $\left|Q_n'(t)\right| \le \dfrac{nM_n}{\sqrt{1-t^2}}$. Then

$$\operatorname*{Var}_{-1}^{1}(Q_n) = \int_{-1}^{1} |Q_n'(t)dt \le nM_n \int_{-1}^{1} (1-t^2)^{-1/2} dt = \pi n M_n \,.$$

Then, based on a semiadditivity property of variations [8], we have

$$\mathrm{Var}(r_{n\mu}) = \mathrm{Var}(f - S_{n\mu}(f)) \le \mathrm{Var}(f) + \mathrm{Var}(S_{n\mu}(f)).$$

By condition, the function f is the bounded variation function, and $S_{n\mu}$ is the polynomial of n degree, thus the inequality (8) can be applied for the assessment of its variation. In case of the pointwise convergence $S_{n\mu}(f)$ to the function f, we can guarantee the boundedness of values $\|S_{n\mu}\|_{C[-1,1]}$, which is uniform with respect to n. The above arguments rove the following lemma.

Lemma 2. Let $S_{n\mu}(f)$ be pointwise converging to the function f. Then

$$\operatorname*{Var}_{-1}^{1}(r_{n\mu}(f)) \le Cn,$$

where C — is the constant, which does not depend on f and n.

A similar statement is true for the variation of the function $r_{n\nu_N}$.

The paper [5] shows that $\rho(\mu,\nu_N) \le C_1 N^{-1-\gamma} + C_2 N^{-1}$, where C_1 и C_2 are constants, which depend on α and β and do not depend on N, $\gamma = \min(\alpha,\beta)$. Based on this inequality and lemma 2, we obtain as follows:

$$\|r_{n\mu}\|^2_{2\nu_N} \le \|r_{n\mu}\|^2_{2\mu} + \rho(\mu,\nu_N) \operatorname*{Var}_{-1}^{1}(r^2_{n\mu}(f)) \le \|r_{n\mu}\|^2_{2\mu} + C' \frac{n}{N^{1+\gamma}} + C'' \frac{n}{N}. \qquad (9)$$

Here, C' и C'' are constants, which do not depend on n and N.

Let us now return to the inequalities of the theorem 1. In this case, the assessment (5) is accurate for the class of functions $L_{\mu\nu}$, but it is inconvenient due to the available function $r_{n\nu_N}$, which has not been theoretically investigated yet. That is why we will turn our attention to the inequality (4). This inequality includes the sum of polynomial squares q_{nN}. For small n values, this sum can be immediately calculated using recurrence relations between the polynomials q_{nN} [4]. For large n values and at $n \to \infty$ we can use asymptotic relations between the polynomials p_n and q_{nN} [3], [9], [10]. Based on these relations, we will further consider conditions of the equiconvergence of the Fourier-Jacobi series and Fourier-Hahn series in the uniform metric for cases $\alpha = \beta = -1/2$ and $\alpha = \beta = 1/2$ (the case $\alpha = \beta = 0$ is investigated in [3]).

It is known [9] that

$$\|p_n - q_{nN}\|_C \le C_3 \frac{n^2}{N}\left(1 + O\left(\frac{n^2}{N}\right)\right) \qquad (10)$$

for kind I Chebyshev polynomials is ($\alpha = \beta = -1/2$), and for kind II Chebyshev polynomials is ($\alpha = \beta = 1/2$)

$$\left\| p_n - q_{nN} \right\|_C \le C_4 \frac{n^3}{N}\left(1+O\left(\frac{n^2}{N}\right)\right), \tag{11}$$

where C_3 and C_4 are universal constants, and the relation of values n and N is conditioned by $n^2/N \to 0$ at $N \to \infty$.

Theorem 3. Let $\alpha = \beta = -1/2$ and $n^2/N \to 0$ at $N \to \infty$. Then $\forall f \in L_{\mu\nu}$

$$\left| R_{nN}(f;t) \right| \le \sqrt{\frac{2n+1}{\pi}\left(\left\| r_{n\mu}(f) \right\|_{2\mu}^2 + C\frac{n}{\sqrt{N}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\right)}\left(1+O\left(\frac{n^2}{N}\right)\right),$$

where C is a universal constant.

Proof. Let $q_{nN}(t) = p_n(t) + \varepsilon_{nN}(t)$. For ε_{nN}, the inequality (10) is satisfied. We shall note than $\varepsilon_{0N}(t) \equiv 0$ [9]. Then

$$\sum_{k=0}^{n} q_{kN}^2 = \sum_{k=0}^{n} (p_k(t)+\varepsilon_{kN}(t))^2 = \sum_{k=0}^{n} p_k^2(t) + 2\sum_{k=1}^{n} p_k(t)\varepsilon_{kN}(t) + \sum_{k=1}^{n} \varepsilon_{kN}(t)^2. \tag{12}$$

For orthonormalized kind I Chebyshev polynomials, the following inequalities are valid [11]:

$$\forall k \ge 1 \quad \left| p_k(t) \right| \le \sqrt{2/\pi}, \quad t \in [-1,1], \quad p_0(t) = \sqrt{1/\pi}.$$

Using these inequalities, as well as inequalities (10), we will majorize the express (12), from where, taking into account the assessment (9), we obtain the theorem statement.

Consequence 1. Let the conditions of theorem 3 be satisfied. Then $\forall f \in C[-1,1]$

$$\left\| R_{nN}(f) \right\|_C \le \sqrt{\frac{2n+1}{\pi}\left(C_1 E_n^2(f) + C_2\frac{n}{\sqrt{N}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\right)}\left(1+O\left(\frac{n^2}{N}\right)\right). \tag{13}$$

where $E_n(f)$ is the error of the best uniform approximation of the function f by a polynomial of the n order, and C_1 и C_2 are universal constants.

Proof. For proofing, it is sufficient to show that $\forall f \in C[-1,1] \quad \left\| r_{n\mu}(f) \right\|_{2\mu} \le CE_n(f)$. Let $Q_n(f;t)$ be the polynomial of the best uniform approximation of the function f. Based on the equation (3), we obtain: $Q_n(f;t) = S_{n\mu}(Q_n(f;t))$. Further, using the proof procedure applied for the known Lebesgue equation [11], we obtain

$$\left\| r_{n\mu}(f) \right\|_{2\mu} \le \left\| f - Q_n(f) \right\|_{2\mu} + \left\| S_{n\mu}(f - Q_n(f)) \right\|_{2\mu}.$$

From the definition of the norm of operators in linear spaces, it follows that

$$\left\| S_{n\mu}(f - Q_n(f)) \right\|_{2\mu} \le \left\| f - Q_n(f) \right\|_{2\mu} \left\| K_{n\mu} \right\|_{2\mu}.$$

Since $\forall f \in L_{2\mu} \quad \left\| r_{n\mu}(f) \right\|_{2\mu} \to 0$ at $n \to \infty$, it follows from the principle of uniform boundedness (Banach-Steinhaus theorem, [12]) that a universal constant $C > 0$ exists, at which $\forall n \quad \left\| K_{n\mu} \right\|_{2\mu} < C$. this assessment, as well as the obvious inequality $\|f-Q_n(f)\|_{2\mu}\le\|f-Q_n(f)\|_C$, where the condition (1) is considered, lead to the proof of the consequence 1.

Consequence 2. Let the conditions of the theorem 3 and $f\in\mathrm{Lip}_\alpha$, $\alpha>½$ be satisfied. Then

$$\|R_{nN}(f)\|_C\to 0 \text{ at } n, N\to\infty.$$

This proposal immediately follows from the assessment (13) and the equation $E_n(f)\le 1/n^\alpha$, which is true for the functions included into the class Lip_α [7].

Theorem 4. Let $\alpha=\beta=1/2$ and $n^2/N\to 0$ at $N\to\infty$. Then $\forall f\in L_{\mu\nu}$ and $\forall t\in[-1,1]$

$$|R_{nN}(f;t)|\le\sqrt{\frac{n(n+2)(2n+5)}{3\pi}\left(\|r_{n\mu}(f)\|_{2\mu}^2+C\frac{n}{\sqrt{N}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\right)}\left(1+O\left(\frac{n^2}{N}\right)\right).$$

For $t\in(-1,1)$

$$|R_{nN}(f;t)|\le\sqrt{\frac{2(n+1)}{\pi(1-t^2)}\left(\|r_{n\mu}(f)\|_{2\mu}^2+C\frac{n}{\sqrt{N}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\right)}\left(1+O\left(\frac{n^2}{N}\right)\right),$$

where C is a universal constant.

The proof procedure is similar to the procedure used for the theorem 3. Here, the inequalities (11) and assessments of norms of the kind II Chebyshev polynomials [11) are used: $\forall t\in[-1,1]$ and for all $n\ge 0$ $|p_n(t)|\le(n+1)\sqrt{2/\pi}$, $\sqrt{1-t^2}\,|p_n(t)|\le 2/\pi$.

Consequence 1. Let the conditions of the theorem 4 be satisfied. Then, if $f\in W^1H^\alpha$ (i. E. The function f is absolutely continuous and $f'\in\mathrm{Lip}_\alpha$), $\alpha>1/2$, then

$$\|R_{nN}(f)\|_C\to 0 \text{ at } n, N\to\infty.$$

But if $f'\in\mathrm{Lip}_\alpha$, $\alpha>1/2$, then $\forall t\in(-1,1)$

$$\|R_{nN}(f;t)\|_C\to 0 \text{ at } n, N\to\infty.$$

The proof procedure is similar to the procedure used for the consequence 2 of theorem 3.

In conclusion, we will give another outcome, which can be applied to the Fourier-Jacobi series and Fourier-Hahn series with arbitrary parameters α and β.

Theorem 5. Let $\alpha>-1$, $\beta>-1$ and $\gamma=\min\{1, 1+\alpha, 1+\beta)$. If at $N\to\infty$ the condition $n/N^\gamma\to 0$ is satisfied, then $\forall f\in L_{\mu\nu}$

$$\|R_{nN}(f)\|_{2\mu}\le C\sqrt{\|r_{n\mu}(f)\|_{2\mu}^2+O\left(\frac{n}{N^\gamma}\right)},$$

where C is a universal constant.

The proof follows from the theorem 2, inequality (9) and uniform boundedness of the norms $\left\| K_{n\nu_N} \right\|_{2\nu_N}$.

The last theorem shows that, contrary to the convergence in the uniform metric, the zero convergence of the difference $R_{nN}(f)$ in the integral metric, exists in any cases, when there is the convergence of $S_{nN}(f)$ to the function f with respect to the space norm $L_{2\mu}$.

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